

# Neutrino mixing in the seesaw model

T.K. Kuo<sup>1,a</sup>, S.-H. Chiu<sup>2,3,b</sup>, G.-H. Wu<sup>4,c</sup>

<sup>1</sup> Department of Physics, Purdue University, West Lafayette, IN 47907, USA

<sup>2</sup> Department of Physics, Rochester Institute of Technology, Rochester, NY 14623, USA

<sup>3</sup> Department of Physics, Frostburg State University, Frostburg, MD 21532, USA

<sup>4</sup> Institute of Theoretical Science, University of Oregon, Eugene, OR 97403, USA

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**Abstract.** In the seesaw model with hierarchical Dirac masses, the neutrino mixing angle exhibits the behavior of a narrow resonance. In general, the angle is strongly suppressed, but it can be maximal for special parameter values. We delineate the small regions in which this happens for the two-flavor problem. On the other hand, the physical neutrino masses are hierarchical, in general, except in a large part of the region in which the mixing angle is sizable, where they are nearly degenerate. Our general analysis is also applicable to the RGE of the neutrino mass matrix, where we find analytic solutions for the running of the physical parameters, in addition to a complex RGE invariant relating them. It is also shown that, if one mixing angle is small, the three-neutrino problem reduces to two two-flavor problems.

## 1 Introduction

The exciting development of recent experiments [1] has offered strong evidence for the existence of neutrino oscillations, from which one can make inferences about the intrinsic properties of the neutrinos. While the neutrino masses (mass differences) are found to be very tiny, there is a major surprise for the mixing angles. It is found that at least one, and possibly two, of the three mixing angles are large, or even maximal. This is in stark contrast to the situation in the quark sector, where all mixing angles are small.

Theoretically, the seesaw model [2] is very appealing in that it can offer a natural mechanism which yields small neutrino masses. However, owing to its complex matrix structure, it is not obvious what the implied patterns of neutrino mixing are. In a previous paper [3], we found a parameterization which enabled us to obtain an exact solution to the two-flavor seesaw model. When one makes the usual assumption that the Dirac mass matrix has a strong hierarchy, the physical neutrino mixing angle exhibits the narrow resonance behavior. For generic parameters in the Majorana mass matrix, the physical neutrino mixing angle is strongly suppressed. However, if the parameters happen to lie in a very narrow region, the mixing can be maximal.

In this paper we will expand on our earlier investigations and discuss in detail the behavior of the neutrino

mixing matrix in the seesaw model. As was shown before, if we assume that the Dirac mass hierarchy is similar to that of the quarks, the problem has three relevant parameters associated with the Majorana sector, namely, the mixing angle, the ratio of the masses, and their relative phase. We will present plots of the physical neutrino mixing angle and their mass ratio in the 3D parameter space. These will offer a bird's-eye view of their behaviors. In particular, the neutrino mixing angle is only appreciable in a very small region, which we exhibit explicitly. Furthermore, this region is complementary to the region in which there is an appreciable physical neutrino mass hierarchy. Thus, roughly speaking, the seesaw model divides the 3D parameter space in two parts. There is a very small region in which the mixing angle is large, while at the same time the neutrino masses are nearly degenerate. For most parameters, the mixing angle is small but there is a strong hierarchy in the mass eigenvalues. An exception to this picture is when the Majorana matrix has an extreme hierarchy and a very small mixing angle. In this tiny region, the physical neutrinos can be hierarchical while simultaneously their mixing angle is large.

The solution to the seesaw problem is most transparent in the parameterization introduced in [3]. However, it is useful to make connections with the usual notation, where individual matrix elements are regarded as independent parameters. We obtain relations which clarify the roles played by the various parameters. They enable one to gain insights in understanding the numerical results presented in the 3D plots.

The general analysis of symmetric and complex matrices turns out to be useful in other applications. Our

<sup>a</sup> e-mail: tkkuo@physics.purdue.edu

<sup>b</sup> e-mail: sxcsps@rit.edu

<sup>c</sup> e-mail: gwu@darkwing.uoregon.edu

method can be used to yield an analytic solution of the renormalization group equation (RGE) of the effective neutrino mass matrix. In addition, we obtain a (complex) RGE invariant which relates the running of the mixing angle and the complex mass ratio. The detailed analysis of the structure of the seesaw matrix also suggests a universal picture for the quark as well as the neutrino mass matrices. While the quarks have generally small mixing angles and hierarchical mass ratios, if one assumes that the Majorana matrix itself is of the seesaw type, the effective neutrino mixing angle can be naturally large.

Finally, we turn to a discussion of the three-neutrino problem. Although the principle involved here is the same as in the two-neutrino problem, the algebra with the Gell-Mann  $\lambda$  matrices is far more complicated than that of the Pauli  $\sigma$  matrices. We are unable to obtain a general solution in this case. However, it is quite well established that one of the neutrino mixing angles is small [4]. In this case, an approximate solution can be obtained. It turns out that, to lowest order, the three-neutrino problem can be reduced to two two-flavor problems. This solution can thus accommodate the ‘‘single-maximal’’ or ‘‘bimaximal’’ solutions that have been considered in the literature.

## 2 The two-flavor problem

In previous papers [3, 5], an exact solution was obtained for the two-flavor seesaw model. In this section, in addition to a summary of the earlier paper, further results will be presented.

For two flavors, the seesaw model,

$$m_\nu = m_D M_R^{-1} m_D^T, \quad (1)$$

can be written in the form

$$m_\nu = U \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} V_R \begin{pmatrix} M_1^{-1} & \\ & M_2^{-1} \end{pmatrix} V_R^T \begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} U^T. \quad (2)$$

Let us introduce the parameterization

$$\begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} = \sqrt{m_1 m_2} e^{-\xi \sigma_3}, \quad \xi = \frac{1}{2} \ln(m_2/m_1); \quad (3)$$

$$\begin{pmatrix} M_1^{-1} & \\ & M_2^{-1} \end{pmatrix} = \sqrt{\frac{1}{M_1 M_2}} e^{2\eta \sigma_3}, \quad (4)$$

$$\eta = \frac{1}{4} \ln(M_2/M_1); \quad (4)$$

$$V_R = e^{i\alpha \sigma_3} e^{-i\beta \sigma_2} e^{i\gamma \sigma_3}. \quad (5)$$

Thus, in the basis in which  $m_D$  is diagonal,  $\beta$  is the mixing angle for  $M_R^{-1}$  while  $\pm 2\gamma$  are the phases of the eigenvalues. This parameterization shows clearly that the relevant variables in the diagonalization of  $m_\nu$  are  $\xi$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\eta$ . Of these, it is usually assumed that  $m_2/m_1$  can be identified with the known quark mass ratio. Also,  $\alpha$  can be absorbed into  $U$  as part of the phase of the Dirac mass

eigenvalues. For  $U \simeq I$ , in particular, it becomes the phase of the charged leptons and is not observable.

Note also that, apart from an overall constant,  $m_\nu$  is a product of  $2 \times 2$  complex matrices with  $\det = +1$ , i.e., it is an element of  $SL(2, C)$ . Thus, we can identify  $m_\nu$  with an element of the Lorentz group, with  $\xi$  and  $\eta$  interpreted as rapidity variables.

To find the effective neutrino mixing matrix, we need to rearrange the matrices in  $m_\nu$  in a different order:

$$m_\nu = \sqrt{\frac{m_1^2 m_2^2}{M_1 M_2}} U W e^{-2\lambda \sigma_3} W^T U^T, \quad (6)$$

$$W = e^{i\omega' \sigma_3} e^{-i\theta \sigma_2} e^{i\phi \sigma_3}, \quad (7)$$

$$\omega' \equiv \omega + \alpha, \quad \lambda = \frac{1}{4} \ln(\mu_2/\mu_1). \quad (8)$$

Here, the physical neutrino masses are given by  $\mu_1$  and  $\mu_2$ , with their ratio given in terms of  $\lambda$ , while  $4\phi$  is their relative phase. We have also absorbed the phase  $\alpha$  into  $\omega'$ . The physical neutrino mixing matrix is given by  $UW$ , so that  $W$  is the induced mixing matrix from the seesaw mechanism. The left-handed Dirac mixing,  $U$ , in analogy to the quark sector, is often taken to be close to the identity,  $U \simeq I$ . In the following we will concentrate on the behavior of  $W$  only, corresponding to  $U \simeq I$ . However, when necessary,  $U$  can always be included in the final result.

As was shown before, the solution for  $W$  corresponds to that of the velocity addition problem in relativity, and one can readily obtain the answer by manipulating the Pauli matrices. We have [6]

$$\tan 2\omega = \frac{\Sigma_I}{\Sigma_R \coth 2\xi - \cos 2\beta}, \quad (9)$$

$$\tan 2\theta = \frac{\sin 2\beta / (\cos 2\omega \cosh 2\xi)}{\cos 2\beta - \Sigma_R \tanh 2\xi - \Sigma_I \tan 2\omega}, \quad (10)$$

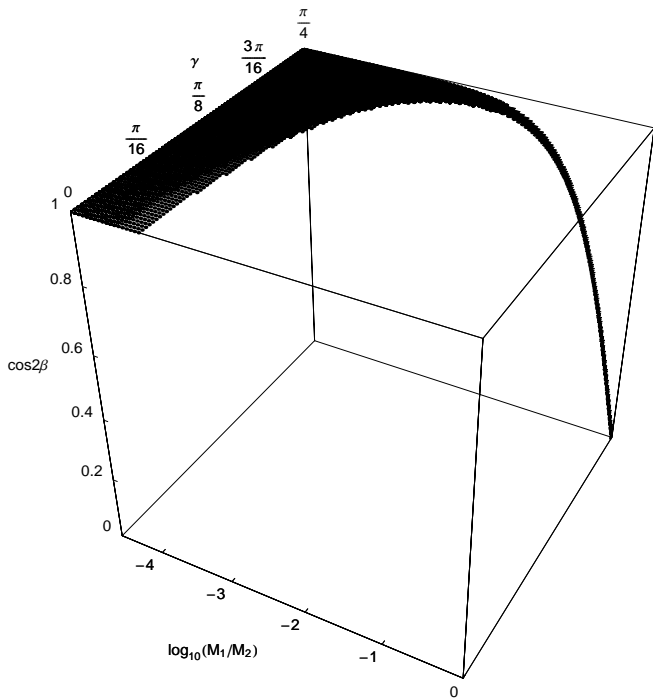
$$\cosh 2\bar{\lambda} = \cosh 2\bar{\xi} \cosh 2\bar{\eta} - \cos 2\beta \sinh 2\bar{\xi} \sinh 2\bar{\eta}, \quad (11)$$

where  $\bar{\lambda} = \lambda + i\phi$ ,  $\bar{\xi} = \xi - i\omega$ ,  $\bar{\eta} = \eta + i\gamma$ , and

$$\begin{aligned} \coth 2\bar{\eta} &= \frac{1 - (M_1/M_2)^2 - 2i(M_1/M_2) \sin 4\gamma}{1 + (M_1/M_2)^2 - 2(M_1/M_2) \cos 4\gamma} \\ &\equiv \Sigma_R + i\Sigma_I. \end{aligned} \quad (12)$$

Note the non-trivial contribution from  $\tan 2\omega$  in (10). To diagonalize the symmetric and complex mass matrix,  $U^{-1} m_\nu U^*$ , as is treated in detail in the next section, it is necessary to multiply the mass matrix on either side by the same phase matrix. This phase matrix is precisely  $e^{-i\omega' \sigma_3}$ .

Equation (10) shows that, when  $m_D$  is hierarchical ( $\xi \gg 1$ ), the neutrino mixing angle  $\theta$  is small ( $\tan \theta \sim e^{-2\xi} \sim m_1/m_2$ ), for generic values of the other parameters,  $\beta$ ,  $\eta$ , and  $\gamma$ . However, when the denominator in (10) vanishes,  $\theta$  is maximal. This is the resonance behavior mentioned before. In general, the seesaw mechanism suppresses the neutrino mixing angle. But when the resonance condition is met, it is enhanced and becomes maximal.

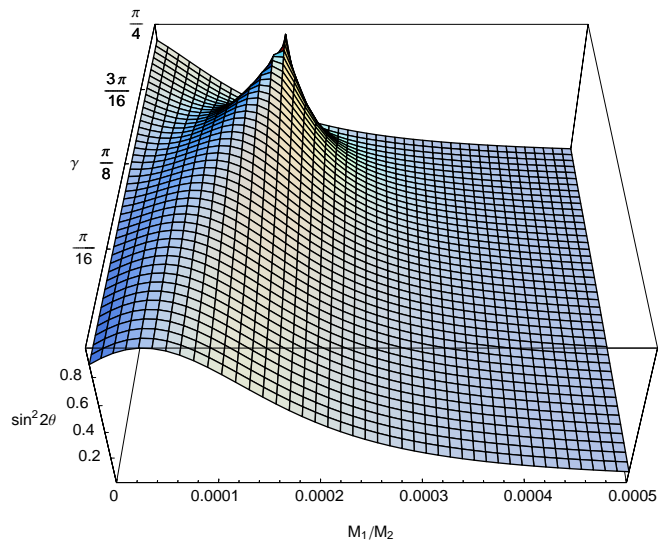


**Fig. 1.** Region in which  $\sin^2 2\theta > 0.5$ , with  $\tanh 2\xi = 0.9998$ , or  $(m_1/m_2) = 0.01$ . Note the log scale used for  $(M_1/M_2)$

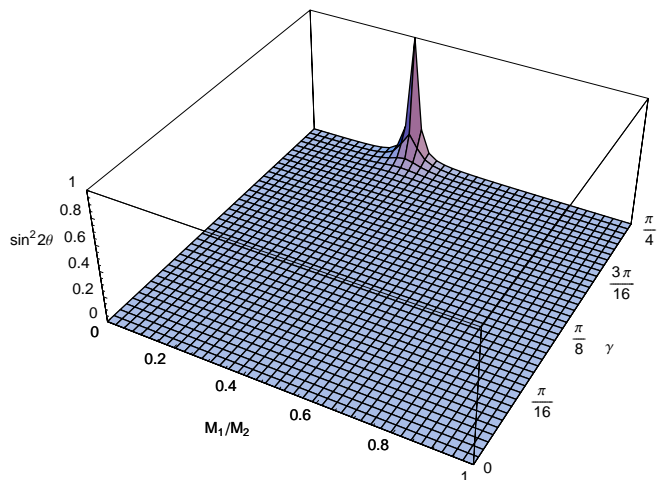
This behavior is quantified in Fig. 1, which is a 3D plot of the region  $\sin^2 2\theta > 0.5$ , within the parameter space spanned by  $\cos 2\beta$ ,  $\gamma$  and  $M_1/M_2$ . This region consists roughly of two parts. One runs along the edge  $\cos 2\beta \approx 1$  and  $M_1/M_2 \ll 1$ , but  $\gamma$  can take values between 0 and  $\pi/4$ . The other region is tube-like, and “hugs” the back wall,  $\gamma \approx \pi/4$ , with  $\cos 2\beta \approx \tanh 2\eta$ . It is striking how small the region for  $\sin^2 2\theta > 0.5$  is. Outside of this region, which consists of most of the parameter space,  $\sin^2 2\theta$  is tiny ( $\sim (m_1/m_2)^2$ ). This result is the analog of the familiar focusing mechanism in relativity. When a relativistic particle decays, most of the decay products are contained in a forward cone of opening angle  $\leq 1/\gamma_0$ , where  $\gamma_0 = 1/(1 - v^2/c^2)^{1/2}$ . This corresponds to the seesaw problem with the identification  $\gamma_0 = \cosh 2\xi \simeq (1/2)(m_2/m_1)$ .

In Fig. 2, we blow up the region with a fixed  $\cos 2\beta \approx 1$ . It is seen that there is considerable structure when  $\sin^2 2\theta$  is maximal. In particular, the dependence on  $\gamma$  is highly non-trivial. From the scale in the figure, we see that large values of  $\sin^2 2\theta$  are confined in a very narrow region with width  $\sim (m_1/m_2)^2$ . Note also that, outside of the maximal  $\sin^2 2\theta$  region, near the upper left edge of Fig. 1 ( $M_1/M_2 \rightarrow 0, \cos 2\beta \rightarrow 1$ ),  $\sin^2 2\theta$  remains large. This region is characterized by an extreme hierarchy of the Majorana masses ( $(m_1/m_2)^2 > (M_1/M_2) \rightarrow 0$ ) and very small  $\beta$  ( $(1 - \cos 2\beta) \sim (m_1/m_2)^2$ ).

Figure 3 shows the contents of Fig. 1 in a 2D parameter space, with  $\cos 2\beta = 1/2$ . It exhibits clearly the behavior of  $\sin^2 2\theta$  near  $\gamma = \pi/4$ . Here, the maximum of  $\sin^2 2\theta$  is attained at  $\cos 2\beta = \tanh 2\eta \cdot \tanh 2\xi$  with  $\gamma = \pi/4$ . Away from these values,  $\sin^2 2\theta$  drops off quickly. The width of the peak is of order  $(m_1/m_2)$  in either  $\Delta(M_1/M_2)$  or  $\Delta\gamma$ .



**Fig. 2.** A plot of  $\sin^2 2\theta$  versus  $(M_1/M_2)$  and  $\gamma$ , with  $\cos 2\beta = 0.9999$ ,  $\tanh 2\xi = 0.9998$ . Note the expanded scale of  $(M_1/M_2)$

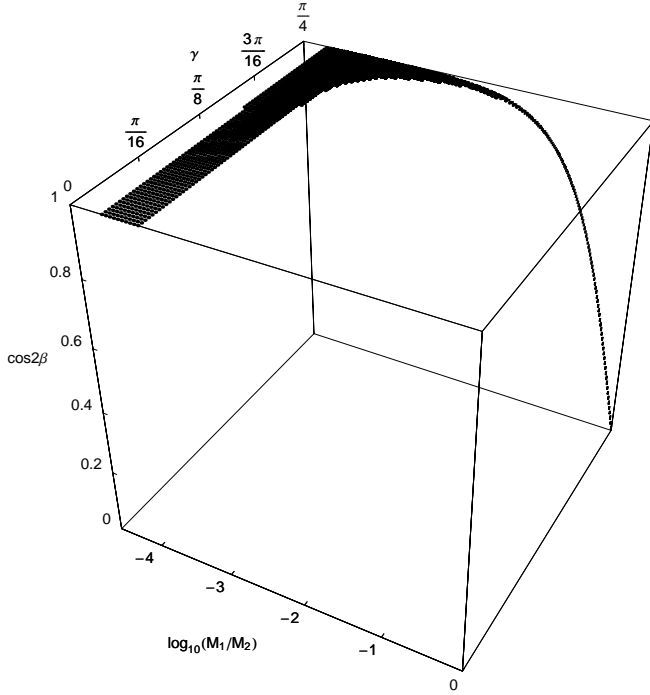


**Fig. 3.** Typical behavior of  $\sin^2 2\theta$  for  $\cos 2\beta < \tanh 2\xi$ . Here,  $\cos 2\beta = 0.5$ ,  $\tanh 2\xi = 0.9998$

The behavior of the physical neutrino mass ratio is depicted in Fig. 4, which exhibits the region of near degeneracy,  $\mu_1/\mu_2 > 0.5$ . We have chosen a log scale for  $M_1/M_2$  to highlight the detailed structure near the upper left edge. A comparison with Figs. 1 and 2 reveals the complementary nature of the regions of maximal  $\sin^2 2\theta$  versus hierarchical  $\mu_1/\mu_2$ . In the small region where  $\sin^2 2\theta \simeq 1$ , one also has  $\mu_1/\mu_2 \simeq 1$ . However, near the upper left edge, for  $(m_1/m_2)^2 > M_1/M_2 \rightarrow 0$ ,  $\mu_1/\mu_2$  can be small and at the same time  $\sin^2 2\theta$  is large. For generic parameters,  $\sin^2 2\theta \sim (m_1/m_2)^2$ , but the masses are also very hierarchical,  $(\mu_1/\mu_2) \sim (m_1/m_2)^2$ .

### 3 General properties of mass matrices

In order to gain some insights about the results presented in the previous section, it is useful to study the general



**Fig. 4.** Region in which the physical neutrino masses are nearly degenerate, with  $\mu_1/\mu_2 > 0.5$ ,  $\tanh 2\xi = 0.9998$

properties of symmetric complex matrices. We will first discuss the relations between different parameterizations of the mass matrices. These relations will shed light on the special properties of matrices of the seesaw type. They will also enable one to gain a qualitative understanding of the results presented in Sect. 2.

### 3.1 Parameterization of neutrino mass matrices

Within the framework of the seesaw model, the neutrinos are Majorana in nature, so that their matrices are symmetric and complex, in general. We first consider the case of two flavors,

$$N = \begin{pmatrix} A & B \\ B & C \end{pmatrix}. \quad (13)$$

Here,  $A$ ,  $B$ , and  $C$  are arbitrary complex numbers. Without loss of generality, we assume that  $N$  is normalized so that  $\det N = +1$ ,

$$AC - B^2 = 1. \quad (14)$$

This matrix can be diagonalized by a unitary matrix  $U$ ,

$$N = U e^{2\eta\sigma_3} U^\dagger. \quad (15)$$

In terms of the eigenvalues  $(n_1, n_2)$ ,  $\eta = (1/4) \ln(n_2/n_1)$ . A convenient choice for  $U$  is the Euler parameterization

$$U = e^{i\alpha\sigma_3} e^{-i\beta\sigma_2} e^{i\gamma\sigma_3}. \quad (16)$$

The relation between the two parameterizations of  $N$  is given by

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} e^{2i\alpha}(\text{ch}2\bar{\eta} + C_{2\beta} \cdot \text{sh}2\bar{\eta}) & S_{2\beta} \cdot \text{sh}2\bar{\eta} \\ S_{2\beta} \cdot \text{sh}2\bar{\eta} & e^{-2i\alpha}(\text{ch}2\bar{\eta} - C_{2\beta} \cdot \text{sh}2\bar{\eta}) \end{pmatrix}, \quad (17)$$

where we have used the notation  $\bar{\eta} = \eta + i\gamma$ ,  $\text{ch}2\eta \equiv \cosh 2\eta$ ,  $S_{2\beta} \equiv \sin 2\beta$ , etc.

Note that because of the condition  $AC - B^2 = 1$ , there are exactly four parameters in the three complex numbers  $A$ ,  $B$ , and  $C$ . To understand the role played by the phase  $\alpha$ , let us write

$$N = \frac{1}{2}(A + C) + \frac{1}{2}(A - C)\sigma_3 + B\sigma_1. \quad (18)$$

The diagonalization of  $N$  is easy provided that the phase of  $A - C$  and  $B$  are the same. In general, we can multiply  $N$  on either side by the same phase,  $e^{-i\alpha\sigma_3}$ ,

$$e^{-i\alpha\sigma_3} N e^{-i\alpha\sigma_3} = \frac{1}{2}(e^{-2i\alpha}A + e^{2i\alpha}C) + \frac{1}{2}(e^{-2i\alpha}A - e^{2i\alpha}C)\sigma_3 + B\sigma_1. \quad (19)$$

We now choose  $\alpha$  so that the phase of  $(e^{-2i\alpha}A - e^{2i\alpha}C)$  coincides with that of  $B$ :

$$\arg B = \arg(e^{-2i\alpha}A - e^{2i\alpha}C). \quad (20)$$

In this case, the matrix  $e^{-i\alpha\sigma_3} N e^{-i\alpha\sigma_3}$  can be diagonalized by  $e^{-i\beta\sigma_2} (e^{-i\alpha\sigma_3} N e^{-i\alpha\sigma_3}) e^{i\beta\sigma_2}$ , with

$$\tan 2\beta = \frac{2B}{e^{-2i\alpha}A - e^{2i\alpha}C} = \text{real}. \quad (21)$$

Thus, given an arbitrary  $N$ , we need first to determine the phase  $\alpha$  by (20). After this  $\beta$  is fixed by (21), and then  $\gamma$  can be read off from the diagonal matrix.

Note that from (21),

$$B(e^{2i\alpha}A^* - e^{-2i\alpha}C^*) = B^*(e^{-2i\alpha}A - e^{2i\alpha}C). \quad (22)$$

Thus, (20) is equivalent to

$$2\alpha = \arg(AB^* + BC^*). \quad (23)$$

In addition, we may use (22) in (21) to obtain

$$\tan 2\beta = \frac{2|AB^* + BC^*|}{|A|^2 - |C|^2}. \quad (24)$$

Note also that, if  $|A| = |C|$ , (20) cannot be used to solve for  $\alpha$ , but (23) and (24) are still valid. The complex eigenvalues of (13) can be obtained from (17). They are given by

$$e^{+2\bar{\eta}} = Ae^{-2i\alpha} + B \tan \beta, \quad (25)$$

$$e^{-2\bar{\eta}} = Ce^{+2i\alpha} - B \tan \beta. \quad (26)$$

It is also useful to introduce another variable,

$$\bar{\zeta} = \frac{1}{2} \ln(A/C). \quad (27)$$

Using  $\bar{\zeta}$ , (21) can be written as

$$\tan 2\beta = \frac{B/\sqrt{AC}}{\text{sh}(\bar{\zeta} - 2i\alpha)}. \quad (28)$$

Also, from (17),

$$\begin{aligned} \text{ch}2\bar{\eta} &= \frac{1}{2}(e^{-2i\alpha}A + e^{2i\alpha}C) \\ &= \sqrt{AC} \cdot \text{ch}(\bar{\zeta} - 2i\alpha). \end{aligned} \quad (29)$$

Similarly,

$$C_{2\beta} \cdot \text{sh}2\bar{\eta} = \sqrt{AC} \cdot \text{sh}(\bar{\zeta} - 2i\alpha). \quad (30)$$

We thus have

$$C_{2\beta} \cdot \tanh 2\bar{\eta} = \tanh(\bar{\zeta} - 2i\alpha). \quad (31)$$

This relation can be regarded as a consistency check on the properties of  $N$ . For instance, if  $C_{2\beta} = 0$  (maximal mixing), it implies that  $\text{Im}\bar{\zeta} = 2\alpha$ , and that  $\text{Re}\bar{\zeta} = 0$ . Another constraint is that the phase of  $\tanh 2\bar{\eta}$  must be the same as that of  $\tanh(\bar{\zeta} - 2i\alpha)$ .

### 3.2 The seesaw transformation

In the seesaw model,  $m_\nu = m_D M_R^{-1} m_D^T$ , it turns out that the properties of  $m_\nu$  is closely related to  $M_R^{-1}$  when we choose a basis in which  $m_D$  is diagonal. We shall call the change from  $M_R^{-1}$  to  $m_\nu$  a ‘‘seesaw transformation’’ (ST). In terms of the notation of the previous section, we define a ST from  $N$  to a new matrix  $M$  by

$$\begin{aligned} M &= e^{-\xi\sigma_3} N e^{-\xi\sigma_3} \\ &= \begin{pmatrix} A' & B' \\ B' & C' \end{pmatrix}. \end{aligned} \quad (32)$$

It is seen immediately that  $B$  and  $AC$  are invariant ( $B' = B$ ,  $A'C' = AC$ ), while

$$A'/C' = e^{-4\xi}(A/C), \quad (33)$$

or

$$\bar{\zeta}' = \bar{\zeta} - 2\xi, \quad (34)$$

where  $\bar{\zeta}' = (1/2) \ln(A'/C')$ . If we assume that

$$M = W e^{2\lambda\sigma_3} W^T, \quad (35)$$

$$W = e^{i(\omega+\alpha)\sigma_3} e^{-i\theta\sigma_2} e^{i\phi\sigma_3}, \quad (36)$$

we can use the results above to derive simple relations between the parameters pertaining to  $M$  and to  $N$ . Thus, from the invariance of  $B$  under ST, we have immediately

$$S_{2\beta} \cdot \text{sh}2\bar{\eta} = S_{2\theta} \cdot \text{sh}2\bar{\lambda}, \quad (37)$$

i.e.,  $S_{2\theta} \cdot \text{sh}2\bar{\lambda}$  is an invariant, independent of  $\xi$ . One of the consequences is that the phase of  $\bar{\lambda}$  is tied to that of  $\bar{\eta}$ , since  $\beta$  and  $\theta$  are both real. In fact, if  $\bar{\lambda} = \lambda + i\phi$ , then

$$\tan 2\phi \cdot \coth 2\lambda = \tan 2\gamma \cdot \coth 2\eta = \text{constant}, \quad (38)$$

independent of  $\xi$ . In particular, if  $\bar{\eta} = \eta + i\pi/4$ ,  $\text{sh}2\bar{\eta}$  is purely imaginary, then the imaginary part of  $\lambda$  must also be  $\pi/4$ , i.e., the mass eigenvalues must have opposite signs. Moreover, given  $\beta$  and  $\bar{\eta}$ , the relation exhibits the complementary nature of  $\theta$  and  $\lambda$ ; large  $\theta$  correlates with small  $\lambda$ , and vice versa. This behavior was already discussed in connection with the results of Fig. 4 in Sect. 2. From (28), the invariance of  $B/(AC)^{1/2}$  yields

$$\tan 2\theta = \tan 2\beta \frac{\text{sh}(\bar{\zeta} - 2i\alpha)}{\text{sh}(\bar{\zeta} - 2\xi - 2i\omega')}. \quad (39)$$

When the ST is hierarchical,  $\xi \gg 1$ , it is clear that, for generic  $\bar{\zeta}$ , the angle  $\theta$  is suppressed ( $\sim 1/\text{sh}2\xi \sim m_1/m_2$ ). However, if  $\zeta \approx 2\xi$ , and if the phases in the denominator of (39) cancel, then  $\theta$  becomes maximal. This was the behavior shown in Fig. 1. It should also be mentioned that (39) reduces to (10) when one uses (31).

As another application, we note that a qualitative understanding of Fig. 1 can be gleaned from (17), (21) and (32). Using (21) and (32), we have

$$\tan 2\theta = \frac{2B}{e^{-2i\omega'}A' - e^{2i\omega'}C'}, \quad (40)$$

with  $\omega = \omega' - \alpha$  given by (9). For  $\xi \gg 1$ , a necessary condition for large  $\theta$  is that  $|C| \simeq 0$  (more precisely,  $|C| \leq e^{-4\xi}|A|$  and  $|C| \leq e^{-2\xi}|B|$ ). From (17), this means that  $\text{ch}2\bar{\eta} \simeq \cos 2\beta \text{sh}2\bar{\eta}$ . However, since  $\text{ch}2\bar{\eta}$  and  $\text{sh}2\bar{\eta}$  have different phases, this equation has only special solutions. They are

(1)  $\gamma = \pi/4$ , so that both  $\text{ch}2\bar{\eta}$  and  $\text{sh}2\bar{\eta}$  are purely imaginary, and  $\cos 2\beta \simeq \coth(2\eta + i\pi/2) = \tanh 2\eta$ . This last equation describes the shaded trajectory on the  $\gamma = \pi/4$  wall in Fig. 1. Another solution is

(2)  $\eta \rightarrow \infty$ , so that  $\text{ch}2\bar{\eta} \simeq \text{sh}2\bar{\eta} \simeq e^{2i\gamma} e^{2\eta}/2$ . Then  $\text{ch}2\bar{\eta} - \cos 2\beta \text{sh}2\bar{\eta} \simeq e^{2i\gamma} e^{2\eta}(1 - \cos 2\beta)/2$ . Since for  $\eta \rightarrow \infty$ ,  $|B| \simeq \sin 2\beta e^{2\eta}$ ,  $\theta$  can be large provided that  $e^{2\xi}(1 - \cos 2\beta) \ll \sin 2\beta$ . This solution corresponds to the shaded region in Fig. 1 with  $\gamma \neq \pi/4$ .

In summary, the neutrino mixing angle  $\theta$  can only be large if the (2, 2) element of  $M_R^{-1}$  is small,  $|C| \simeq 0$ . The precise value depends on phases and possible cancellation between  $A'$  and  $C'$ . Note that in the literature a number of studies has concentrated on the case of  $M_R^{-1}$  being a real matrix. For large mixing, two types of  $M_R^{-1}$  have been identified:

$$(1) \quad M_R^{-1} \simeq \begin{pmatrix} 1 & \epsilon \\ \epsilon & \epsilon^2 \end{pmatrix};$$

$$(2) \quad M_R^{-1} \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\text{or} \quad M_R^{-1} \simeq i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so that  $\det M_R^{-1} = +1$ . These are special cases of  $|C| \simeq 0$ , corresponding to the two end points of the shaded region in Fig. 1, with coordinates  $(M_1/M_2, \cos 2\beta, \gamma) \simeq (0, 1, 0)$  and  $(1, 0, \pi/4)$ , respectively. Our analysis shows that care must be taken when we have small deviations from these forms, which arise naturally in models constructed from a presumed broken symmetry. The narrowness of the shaded region means that a viable solution can be easily thrown off course by small perturbations. An example of such sensitivities is known in the renormalization group running effects, which we will discuss in the next section.

### 3.3 Renormalization

It turns out that our general analysis has an immediate application to the renormalization group equation (RGE) analysis of the neutrino mass matrix. We briefly comment on this connection. A full account will be given elsewhere [7].

In the SM and MSSM, the RGE running of the neutrino mass matrix has been very extensively studied [8, 9]. For simplicity, we only consider the two-flavor problem with  $(\nu_\mu, \nu_\tau)$ . The RGE for the effective neutrino mass matrix is given by

$$\frac{d}{dt} m_\nu = -(\kappa m_\nu + m_\nu P + P^T m_\nu), \quad (41)$$

where  $\kappa$  is related to the coupling constants,  $t = 1/(16\pi^2) \ln \mu/M_X$ , and to a good approximation

$$P \simeq P^T \simeq \chi(1 - \sigma_3), \quad (42)$$

where  $\chi$  is given by  $y_\tau^2/4$  in the SM and  $-\tilde{y}_\tau^2/2$  in the MSSM, with  $y_\tau$  and  $\tilde{y}_\tau$  being the  $\tau$  Yukawa coupling in the SM and MSSM respectively. The solution to RGE is

$$m_\nu(t) = e^{-\kappa' t} e^{\xi \sigma_3} m_\nu(0) e^{\xi \sigma_3}, \quad (43)$$

where  $\kappa' = \kappa + 2\chi$ ,  $\xi = \chi t$ , and we have ignored the  $t$ -dependence of the coupling constant so that  $\int \kappa dt \simeq \kappa t$ , etc.

It is convenient to factor out the determinant

$$m_\nu = \sqrt{m_1 m_2} M. \quad (44)$$

Then

$$\sqrt{m_1(t) m_2(t)} = e^{-\kappa' t} \sqrt{m_1(0) m_2(0)}, \quad (45)$$

$$M(t) = e^{\xi \sigma_3} M(0) e^{\xi \sigma_3}. \quad (46)$$

Thus, while the overall scale  $(m_1 m_2)^{1/2}$  has a simple exponential dependence on  $t$ , the running of  $M$ , which contains the mass ratio and the mixing angle, is just a seesaw transformation defined in the previous section. The difference from the traditional seesaw model is that, instead of  $\xi \gg 1$ , for the RGE the running  $\xi$  is usually small ( $\sim 10^{-5}$  in the SM). Nevertheless, the exact and analytic formulae given in (9)–(12) are valid solutions of the RGE. A

detailed analysis of their properties will be given in a separate paper. We only note that, according to (37), there is a (complex) RGE invariant,

$$\sin 2\theta(t) \sinh 2\bar{\lambda}(t) = \sin 2\theta(0) \sinh 2\bar{\lambda}(0), \quad (47)$$

where  $\theta$  and  $e^{4\bar{\lambda}}$  ( $\bar{\lambda} = \lambda + i\phi$ ) are, respectively, the physical neutrino mixing angle and the mass ratio. This equation can be used to determine  $\lambda(t)$  and  $\phi(t)$ , once  $\theta(t)$  is obtained from (10). We should also emphasize that, because the solution can exhibit resonant behavior, a large effect can result even for very small running  $\xi \ll 1$ .

### 3.4 Parameterization of the three-flavor matrix

As is clear from the previous discussions, the Euler parameterization is the most convenient one for dealing with the two-flavor problem. The generalization to three flavors, then, amounts to parameterizing an  $SU(3)$  element in the form (phase)(rotation)(phase). However, there are altogether eight parameters in  $SU(3)$  while each phase matrix can only accommodate two. So there must also be an additional phase matrix contained in the rotational part of a general  $SU(3)$  matrix. This decomposition is of course none other than the familiar CKM matrix decomposition. Thus, for three flavors, the analog of (5) is

$$V_R = e^{i(\varepsilon_3 \lambda_3 + \varepsilon_8 \lambda_8)} e^{i\varepsilon_7 \lambda_7} e^{i\varepsilon_5 \lambda_5} e^{i\delta_3 \lambda_3} e^{i\varepsilon_2 \lambda_2} e^{i(\varepsilon'_3 \lambda_3 + \varepsilon'_8 \lambda_8)}. \quad (48)$$

Like the CKM representation, the phase factor  $e^{i\delta_3 \lambda_3}$  could be put in a different location, or one could use another diagonal  $\lambda$  matrix.

The seesaw problem for three flavors again aims at rewriting the matrices so that  $m_\nu$  is given as in (6), with  $W$  assuming the form of (48). As in the two-flavor case, the exterior phase factors of  $W$  do not contribute to neutrino oscillations. An exact solution for the three-flavor problem, however, is not easily obtained owing to the complexity of computing finite matrices involving the  $\lambda$  matrices. In Sect. 5, we will present an approximate solution to the three-neutrino problem.

## 4 A unified approach to fermion mass matrices

Our general analysis of the properties of the seesaw model suggests a unified picture of the quark and neutrino mass matrices. As was discussed in Sect. 2, the physical mixing angle of a seesaw model is quite small, in general, but can be maximal when special conditions are met. We will now present arguments which can associate these regions to the quark and neutrino mass matrices, respectively. For simplicity, our discussions are restricted to the case of two flavors only.

#### 4.1 Quark mass matrices

It has been known for a long time that quark mass matrices can be adequately described in a seesaw form [10, 11]

$$m = \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix}, \quad (49)$$

where  $a, b, c$  are arbitrary complex numbers, all of the same order, and  $\mu_2/\mu_1 \gg 1$ . Here we have used the arbitrariness in  $m$  to demand that it be complex and symmetric, in contrast to the usual choice that  $m$  is hermitian.

If the physical masses are denoted as  $m_1$  and  $m_2$ , then (49) implies that, for generic values of  $a, b, c$ ,

$$m_1/m_2 \sim (\mu_1/\mu_2)^2. \quad (50)$$

The mixing angle satisfies the well-known relation [10]

$$\sin^2 \theta \simeq m_1/m_2. \quad (51)$$

This result was derived first for real matrices, and remains valid for the complex case, as discussed in Sect. 2. It has served as a model for quark mass matrices for a long time.

Physically, a symmetric and complex mass matrix can be derived by a symmetry argument. Since the mass term in the lagrangian is given by  $\bar{q}_L M q_R$ , a symmetric mass matrix can be naturally obtained by imposing a discrete  $Z_2$  symmetry:

$$\bar{q}_L \leftrightarrow q_R. \quad (52)$$

If we further impose a gauged horizontal symmetry, such as a  $U(1)$  symmetry à la Froggatt and Nielsen [11], then we are led to a mass matrix in the form of (49). For instance, we may take the horizontal charge assignments  $(0, 1)$  for  $(q_{L1}, q_{L2})$  and  $(1, 0)$  for  $(q_{R1}, q_{R2})$ . The charge assignments for the mass matrix  $\bar{q}_L M q_R$  is

$$Q_M \sim \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}. \quad (53)$$

The Froggatt–Nielsen mechanism then calls for a mass matrix of the form

$$M \sim \begin{pmatrix} \epsilon^2 a & \epsilon b \\ \epsilon b & c \end{pmatrix}, \quad (54)$$

as in (49).

#### 4.2 Neutrino mass matrices

In Sect. 3.2, we have found that, in order to have a large physical mixing angle, it is necessary that the  $(2, 2)$  element of  $M_R^{-1}$  be small, i.e.,  $|C| \simeq 0$  for

$$M_R^{-1} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}.$$

Since we have used the normalization  $\det M_R^{-1} = +1$ , the Majorana mass matrix is given by

$$M_R = \begin{pmatrix} C & -B \\ -B & A \end{pmatrix}. \quad (55)$$

The condition  $|C| \simeq 0$  simply means that  $M_R$  is itself of the seesaw form. The condition  $|C| \simeq 0$  is not sufficient, however, to guarantee a large mixing angle, which is a consequence of further constraints on  $M_R$ . We will not attempt a detailed model construction here. We only note that, as emphasized in Sects. 2 and 3, the mixing angle is very sensitive to a small variation of the parameters in  $M_R$ . In particular, if a model is based on symmetry arguments, symmetry breaking effects have to be weighted carefully.

In summary, both the quark and neutrino mass matrices can be adequately described in the seesaw form. Their difference arises from the Majorana sector, which is itself of the seesaw form. This last requirement can lead to large mixing in the effective neutrino mass matrix. The sensitivity to small changes in the parameters calls for a careful examination which should also include three-flavor effects. Detailed model construction along these lines will be attempted in the future.

### 5 An approximate solution to the three-flavor problem

In Sect. 3.4, it was pointed out that the three-flavor seesaw [12] problem amounts to rearranging products of matrices in  $SL(3, C)$ . Since a general analytical solution is not available, we will turn to an approximate solution which is physically relevant.

For the three-neutrino problem, it is known that the (23) angle is near maximal, the (13) angle is small, and that the (12) angle is probably large. This suggests that, to a good approximation, the three-flavor problem can be decomposed into two two-flavor problems. To implement this scenario, let us consider the  $3 \times 3$  matrix  $M_R^{-1}$ ,

$$M_R^{-1} = \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix}. \quad (56)$$

The neutrino matrix, with  $m_D$  diagonal and  $U = I$  for simplicity of presentation, since the general case can be easily incorporated as in (2), is given by

$$m_\nu = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix}. \quad (57)$$

It is convenient to introduce, in addition to the Gell-Mann  $\lambda$  matrices,  $\lambda_9$  and  $\lambda_{10}$ ,

$$\lambda_9 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad (58)$$

$$\sqrt{3}\lambda_{10} = \begin{pmatrix} -2 & & \\ & 1 & \\ & & 1 \end{pmatrix}. \quad (59)$$

We may now write

$$\begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} = \begin{pmatrix} m_1 & & \\ & \sqrt{m_2 m_3} & \\ & & \sqrt{m_2 m_3} \end{pmatrix} e^{-\xi \lambda_9}, \quad (60)$$

$$\xi = \frac{1}{2} \ln(m_3/m_2). \quad (61)$$

Then, the (23) submatrix of  $m_\nu$  can be diagonalized,

$$e^{-\xi \lambda_9} \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} e^{-\xi \lambda_9} = U \begin{pmatrix} A & B' & D' \\ B' & A' & 0 \\ D' & 0 & \Sigma' \end{pmatrix} U^T, \quad (62)$$

$$U = e^{i\alpha \lambda_9} e^{i\beta \lambda_7} e^{i\gamma \lambda_9}, \quad (63)$$

$$e^{-\xi \lambda_9} \begin{pmatrix} A \\ B \\ D \end{pmatrix} = U \begin{pmatrix} A \\ B' \\ D' \end{pmatrix}, \quad (64)$$

and  $(A', \Sigma')$  are the eigenvalues. Although we could have chosen a proper normalizing factor so that the (23) submatrix has  $\det = \pm 1$ , as in Sect. 2, for this problem it is simpler not to do this and  $A' \Sigma' \neq 1$ , in general. If we absorb  $e^{i\gamma \lambda_9}$  by defining the new variables  $(B'', D'', A'', \Sigma'') = (e^{i\gamma} B', e^{-i\gamma} D', e^{2i\gamma} A', e^{-2i\gamma} \Sigma')$ , and since  $\lambda_7$  and  $\lambda_9$  commute with the remaining Dirac matrix, (57) becomes

$$m_\nu = X \begin{pmatrix} A & B'' & D'' \\ B'' & A'' & 0 \\ D'' & 0 & \Sigma'' \end{pmatrix} X^T, \quad (65)$$

$$X = e^{i\alpha \lambda_9} e^{i\beta \lambda_7} \begin{pmatrix} m_1 & & \\ & \sqrt{m_2 m_3} & \\ & & \sqrt{m_2 m_3} \end{pmatrix}. \quad (66)$$

Now, the (13) rotation is controlled by  $|D''/\Sigma''|$ . However, we must first make sure that they have the same phase (with the approximation  $|m_1^2 A| \ll |m_2 m_3 \Sigma''|$ ). To this end let us multiply  $m_\nu$  by  $e^{i\omega \sqrt{3} \lambda_{10}}$  on either side, and choose  $\omega$  so that  $e^{-i\omega} D''$  and  $e^{2i\omega} \Sigma''$  have the same phase,  $\arg(e^{-i\omega} D'') = \arg(e^{2i\omega} \Sigma'')$ . In this case, we can rotate away the (13) element of  $m_\nu$  without changing its other elements by assuming that the angle of rotation is small,  $|m_1 D''| \ll |\sqrt{m_2 m_3} \Sigma''|$ . We have (with  $A'' = A - D''^2/\Sigma''$ ) approximately

$$m_\nu \simeq Y \begin{pmatrix} A'' & B'' & 0 \\ B'' & A'' & 0 \\ 0 & 0 & \Sigma'' \end{pmatrix} Y^T, \quad (67)$$

$$Y = e^{i\alpha \lambda_9} e^{-i\omega(\sqrt{3} \lambda_{10})} e^{i\beta \lambda_7} e^{-i\psi \lambda_5} e^{i\omega(\sqrt{3} \lambda_{10})} \times \begin{pmatrix} m_1 & & \\ & \sqrt{m_2 m_3} & \\ & & \sqrt{m_2 m_3} \end{pmatrix}, \quad (68)$$

$$\tan \psi = (m_1 e^{-i\omega} D'') / (\sqrt{m_2 m_3} e^{2i\omega} \Sigma'') = \text{real}. \quad (69)$$

After this somewhat laborious route, we see that the diagonalization of  $m_\nu$  can be finally achieved by working solely in the (12) sector. The crucial assumption for the success of this procedure is that  $\tan \psi \ll 1$ . Otherwise the (13) rotation  $e^{-i\psi \lambda_5}$  will generate non-negligible elements all over the matrix  $m_\nu$ . Although the exact condition for  $\tan \psi \ll 1$  seems complicated, in practice, as long as the elements  $B$  and  $D$  in  $M_R^{-1}$  are reasonably small, the approximation is valid.

Fortunately, it is known that in reality the physical (13) rotation angle is small. This means that for any successful  $m_\nu$ , the above approximation is appropriate. In this case, the three-neutrino problem is reduced to two two-flavor problems. In particular, two popular scenarios, the bimaximal or single-maximal models, can be accommodated.

## 6 Conclusion

Recent experimental data have revealed two striking features of the intrinsic properties of the neutrinos. One, as expected, they are very light. Two, perhaps surprisingly, at least some of their mixing angles are large, or even maximal. The seesaw model provides a natural explanation of the lightness. However, the story of the mixing angles is more complicated. In the seesaw model, the neutrino mixing matrix can be written as  $UW$ , where  $U$  comes from the left-handed rotation which diagonalizes the Dirac mass matrix, and  $W$ , defined in (7), is induced from the right-handed sector of the model. For two flavors, the analytic solution for  $W$  shows that, when there is a mass hierarchy in  $m_D$ , the mixing angle in  $W$  is greatly suppressed for most of the available parameter space. However, in a very small region, which we exhibited explicitly in Sect. 2, the mixing angle can be large. In addition, this region may be divided roughly into two parts. In one, characterized by  $\gamma \approx \pi/4$ , the physical neutrino masses are nearly degenerate. In the other, in which the Majorana mass eigenvalues are hierarchical, the neutrino masses can be either hierarchical or nearly degenerate. This behavior of  $W$  has interesting theoretical implications.

Since the neutrino mixing matrix is given by  $UW$ , there are three obvious possibilities which can lead to large mixing.

- (1)  $U$  contains large angles but  $W \simeq I$ ;
- (2) both  $U$  and  $W$  contribute appreciably and they add up to form large mixing;
- (3)  $U \simeq I$  but the large angle is in  $W$ . Corresponding to these possibilities we have three different physical scenarios. With  $W \simeq I$ , case (1), the physical neutrino masses are highly hierarchical. The burden for the model builders is to find a credible theory which makes  $U$  almost maximal naturally. The scenario of case (2) seems the least likely to be implemented. This is a “just-so” solution whereby the Dirac and Majorana sectors must conspire to make the resultant angle large. In case (3),  $U \simeq I$  is quite reasonable from quark-lepton symmetry, which leads natu-



rally to  $U \sim U_{CKM}$ . The challenge is to find a mechanism whereby the parameters in the seesaw model lie naturally in the narrow range for large mixing.

In Sect. 4.2, we have identified a necessary condition for large mixing, namely, that the Majorana mass matrix is also of the seesaw type. This result suggests a universal seesaw mechanism for both the quark and neutrino mass matrices. The quarks can take advantage of the general solution, resulting in small mixing and hierarchical masses. For neutrinos, the existence of  $M_R$  can then lead to large mixing angles. More detailed studies are necessary to implement this scenario.

Our general analysis of symmetric and complex matrices also has an immediate application to the RGE running of the neutrino mass matrices. Exact and analytic solutions of the RGE are found, in addition to a (complex) RGE invariant which relates explicitly the running of the mixing angle, the mass ratio and its phase.

The analyses given above are for the case of two flavors. However, in the approximation of a small ( $13^\circ$ ) angle, we have found that the three-flavor problem is reduced to two two-flavor problems. We thus do not expect qualitatively different physics for this case.

In conclusion, the neutrino mixing matrix (masses and mixing angles) implied by the seesaw model has rather intriguing properties. To accommodate large mixing angles, there are just a few limited options available. These conditions should be helpful in the search of a viable neutrino mass matrix. We hope to return to this topic in the future.

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## References

1. Super Kamiokande Collaboration, Y. Fukuda et al., Phys. Rev. Lett. **82**, 2644 (1999)
2. M. Gell-Mann, P. Ramond, R. Slansky, in Supergravity, edited by P. van Nieuwenhuizen, D. Freedman (North Holland, Amsterdam 1979); T. Yanagida, in Proceedings of the Workshop on Unified Theory and Baryon Number in the Universe, edited by O. Sawada, A. Sugamoto (KEK, 1979)
3. T.K. Kuo, G.-H. Wu, S.-H. Chiu, Phys. Rev. D **62**, 051301 (2000)
4. CHOOZ Collaboration, M. Appollonio et al., Phys. Lett. B **420**, 397 (1998)
5. For earlier studies of the two-flavor problem, see A. Smirnov, Phys. Rev. D **48**, 3264 (1993); M. Tanimoto, Phys. Lett. B **345**, 477 (1995)
6. See (23), (25), (26) and (27) of [3]
7. T.K. Kuo, J. Pantaleone, G.-H. Wu, hep-ph/0104131
8. K. Babu, C.N. Leung, J. Pantaleone, Phys. Lett. B **319**, 191 (1993), hep-ph/9309223; P.H. Chankowski, Z. Plucienik, Phys. Lett. B **316**, 312 (1993), hep-ph/9306333
9. J. Ellis, S. Lola, Phys. Lett. B **458**, 310 (1999), hep-ph/9904279; J.A. Casas, J.R. Espinosa, A. Ibarra, I. Navarro, Nucl. Phys. B **573**, 652 (2000), hep-ph/9910420; *ibid*, B **569**, 82 (2000), hep-ph/9905381; P.H. Chankowski, W. Krolikowski, S. Pokorski, Phys. Lett. B **473**, 109 (2000), hep-ph/9910231; N. Haba et al., Prog. Theor. Phys. **103**, 145 (2000), hep-ph/9908429; S.F. King, N. Nimai Singh, Nucl. Phys. B **591**, 3 (2000), hep-ph/0006229; Z.-Z. Xing, Phys. Rev. D **63**, 057301 (2001), hep-ph/0011217; K.R.S. Balaji, R.N. Mohapatra, M.K. Parida, E.A. Paschos, hep-ph/0011263
10. S. Weinberg, Trans. NY Acad. Sci. (Ser. II) **38**, 185 (1977); A. Davidson, K.C. Wali, Phys. Rev. Lett. **59**, 393 (1987)
11. C.D. Froggatt, H.B. Nielsen, Nucl. Phys. B **147**, 277 (1979)
12. For reviews of the neutrino mass matrix problem, see, e.g., G. Altarelli, F. Feruglio, Phys. Rep. **320**, 295 (1999); S.M. Barr, I. Dorsner, Nucl. Phys. B **585**, 79 (2000)